CERTAIN EQUIVALENT VALUE FORMULAS
for
DECISION ANALYSIS

INFORMS/CORS NATIONAL CONFERENCE
MONTREAL - April 26 – 29, 1998

Ronald E. Davis, Ph.D.
Marketing, MIS, and Decision Sciences Dept.
San Jose State University
San Jose, California

http://www.matpro.com
rondavis@matpro.com

PRESENTATION OUTLINE

EXPONENTIAL UTILITY

UTILITY FUNCTION

$$U(x) = 1 - EXP(-x/\tau)$$

where x and τ are both measured in dollars (x is wealth OR increment to wealth).

RISK TOLERANCE (τ)

Your risk tolerance τ for a given gamble is (approximately) twice your answer to the following question (at the time you face the given gamble):

> WHAT IS THE LARGEST AMOUNT YOU WOULD WAGER ON A TRIPLE-OR-NOTHING BET WITH EVEN ODDS?

NOTE: RISK TOLERANCE MAY VARY WITH WEALTH.

As wealth increases, risk tolerance generally increases as well.

CE VALUE CONCEPT DEFINITION

OPERATIONAL/EMPIRICAL

Consider a gamble G defined by a finite collection of possible monetary payoffs (x_i) with associated probabilities (p_i):

$G = \{x_i, p_i\}_{i=1-n}$

or defined by a probability density function f(x).

IF POSITIVE VALUED (preferred to status quo):

What is the least amount of cash to be received in an envelope which you would accept in lieu of the gamble G?

IF NEGATIVE VALUED (prefer status quo):

What is the most amount of cash you would pay in an envelope in lieu of taking the gamble G?

In the first cast, CE = "Answer";
In the second case, CE = -"Answer"

THEORETICAL

If the utility for the decision maker is a known function U(·), what is the value of x, call it $\tilde{x}$ such that:

$$U(\tilde{x}) = \sum_{i=1}^{n} p_i U(x_i) \quad or \quad = \int_{-\infty}^{\infty} f(x)U(x)dx$$

or

$$\tilde{x} = U^{-1}(\sum_{i=1}^{n} p_i U(x_i)) \quad or \quad = U^{-1}(\int_{-\infty}^{\infty} f(x)U(x)dx)$$

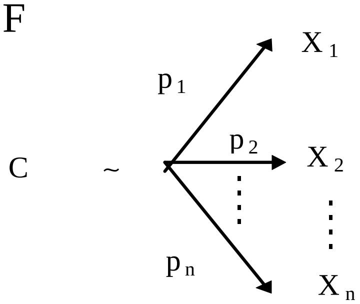
VALUE ADDITIVITY PROPERTY

A utility function U(x) has the Value Additivity property if:

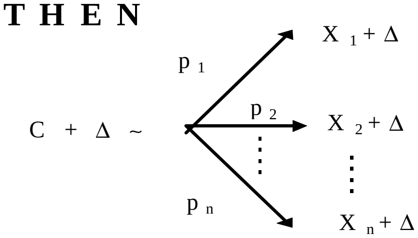
$$CE(L_1 \,\&\, L_2) = CE(L_1) + CE(L_2)$$

for all independent gambles L_1 and L_2. If L2 is a cash amount, then in the finite case one has the familiar Δ-Property,

IF



T H E N



THEOREMS

The Δ-Property and the Value Additivity property are equivalent.

U(·) has the Value Additivity property ⇔ U(·) is Linear or Exponential

Ron Davis, “Decision Policy Optimization via Certain Equivalent Functions for Exponential Utility” (JAI Press, Advances In Mathematical Programming and Financial Planning, 1997)

CERTAIN EQUIVALENCE FUNCTIONS

DEFINITION

A mapping from the parameters of a probability distribution for monetary payoff and a risk tolerance value to a certain equivalent dollar value (via the exponential utility function):

$$CE_\tau = \$\text{-VALUE}(\tau, par_1, par_2, \ldots, par_m)$$

$$= U^{-1}(E[U(x)]) = -\tau \ln(E[EXP(-x/\tau)])$$

APPLICATION

If Math Model maps decision variables to payoff distribution parameters

$$M: (d_1, d_2, \ldots, d_n) \text{-->} (par_1, par_2, \ldots, par_m)$$

and CE_τ maps payoff distribution parameters to \$-Value, can formulate decision problem as follows:

CHOOSE decision variables $(d_1, d_2, \ldots, d_n)$ in such a way as to

MAXIMIZE CE_τ

DERIVATION OF CERTAIN EQUIVALENT FORMULAS

Choose certainty equivalent for a probability distribution with density f(x) as that value $X_{\equiv}$ such that

$$e^{-x_{\equiv}/\tau} = \int_{-\infty}^{\infty} f(x) e^{-x/\tau} dx \quad or \quad = \sum p_i e^{-x_i/\tau}$$

or, solving for $X_{\equiv}$, one obtains

$$X_{\equiv} = -\tau \ln(\int_{-\infty}^{\infty} f(x) e^{-x/\tau} dx) \quad or \quad = -\tau \ln(\sum p_i e^{-x_i/\tau})$$

If a distributional family is parameterized by (α_1, …, α_m) then the associated CE-Value function is denoted

$$CE_{\tau}(\alpha_1, \ldots, \alpha_m)$$

SOME KNOWN CERTAIN EQUIVALENT FORMULAS

Some distributions admit closed form formula CE functions:

FINITE DISCRETE LOTTERY $\{x_i, p_i\}_{i=1...n}$

$$\tilde{x} = -\tau \ln\left(\sum_{i=1}^{n} p_i EXP(-x_i \,/\, \tau) \right)$$

NORMAL DISTRIBUTION (μ,σ)

$$\tilde{x} = \mu - \sigma^2 \,/\, (2\tau)$$

UNIFORM DISTRIBUTION (A,B)

$$\tilde{x} = \frac{A+B}{2} - \tau \ln\left[\frac{EXP((B-A)/(2\tau)) - EXP(-(B-A)/(2\tau))}{(B-A)/\tau} \right]$$

$$= \frac{A+B}{2} - \tau \ln\left[\frac{\sinh((B-A)/(2\tau))}{(B-A)/(2\tau)} \right]$$

HISTOGRAM DISTRIBUTION $(\langle x_0, x_1, ..., x_n \rangle, \langle p_1, p_2, ..., p_n \rangle)$:

$$\tilde{x} = -\tau \ln\left\{ \sum_{i=1}^{n} p_i EXP(-(x_{i-1} + x_i)/(2\tau)) \left(\frac{\sinh((x_i - x_{i-1})/(2\tau))}{(x_i - x_{i-1})/(2\tau)} \right) \right\}$$

GAMMA DISTRIBUTION $(\underline{x}, \mu, \sigma)$

$$\tilde{x} = \underline{x} + \tau \frac{(\mu - \underline{x})^2}{\sigma^2} \ln\left[1 + \frac{\sigma^2}{\tau(\mu - \underline{x})} \right]$$

TABLE 2A: DISCRETE DISTRIBUTIONS

NAME	PARAMETERS	FORMULA/SERIES
Finite Discrete	$\{x_i,p_i\}$i=1...n	$-\tau \ln(\sum p_i \exp(-x_i/\tau))$ where $\sum p_i = 1$
Binomial	n, p (q = 1-p)	$-n\tau \ln(q + p \exp(-1/\tau))$
Geometric	q (p = 1 - q)	$\tau \ln((1 - q \exp(-1/\tau))/p)$
HyperGeometric	N, R, n	Finite Discrete with Hyper-Geometric probabilities
Negative Binomial	n, p	$n\tau \ln((1 - q \exp(-1/\tau))/p)$
Pascal	r, p (q = 1 - p)	$r\tau \ln((1 - q \exp(-1/\tau))/(p \exp(-1/\tau)))$
Poisson	μ	$\mu\tau (1 - \exp(-1/\tau))$
Uniform - Discrete	n_a, n_b	$n_a - \tau \ln\left(\frac{1-e^{-(n_b-n_a+1)/\tau}}{(1-e^{-1/\tau})(n_b-n_a+1)}\right)$

TABLE 2B: CONTINUOUS DISTRIBUTIONS

NAME	PARAMETERS	FORMULA/SERIES
beta	A, B, α, β	B - τ ln($\sum a_k$) where $a_0 = 1$, $a_k = a_{k-1} * \frac{(\beta + k + 1)(B - A)/\tau}{k(\alpha + \beta + k - 1)}$
Chi-Square	n, σ	$(n\tau/2)\ln(1 + 2\sigma^2/\tau)$
Erlang	k, λ	$k\tau \ln(1 + 1/(k\lambda\tau))$
Exponential	μ	$\tau \ln(1 + \mu/\tau)$
gamma (shifted to A)	A, μ, σ	$A + \frac{\tau(\mu - A)^2}{\sigma^2}\ln\left(1 + \frac{\sigma^2}{\tau(\mu - A)}\right)$
Histogram - continuous	$x_0,p_1,x_1,p_2,\ldots,p_n,x_n$	$-\tau \ln\left[\sum_1^n p_i e^{-m_i/\tau}\left(\frac{\sinh(w_i/\tau)}{(w_i/\tau)}\right)\right]$ where $m_i=(x_{i-1}+x_i)/2$ $w_i=(x_i-x_{i-1})/2$
Laplace	μ, σ ($\tau>\sigma$/sqrt(2))	$\mu + \tau \ln(1 - \sigma^2/(2\tau^2))$
Logistic	μ, β	numerical integration
LogNormal	μ, σ	numerical integration
Normal	μ, σ	$\mu - 0.5\,\sigma^2/\tau$
Rayleigh	a	numerical integration
Triangular	a, m, b	$a - \tau \ln\left\{\left(\frac{2\tau}{b-a}\right)\left[\frac{\left(1 - e^{-(m-a)/\tau}\right)}{(m-a)/\tau} - e^{-(m-a)/\tau}\frac{\left(1 - e^{-(b-m)/\tau}\right)}{(b-m)/\tau}\right]\right\}$
Uniform – continuous	a, b	$\mu - \tau \ln\left(\frac{\sinh(w/\tau)}{(w/\tau)}\right)$ where $\mu=(a+b)/2$ and $w = (b-a)/2$
Weibull	a, b	numerical integration

EVPI Concept Extension

Normal Form Analysis

		ACTIONS				
probability	**State**	A1	A2	A3	A4	v*
p1	θ1	v11	v12	v13	v14	v1*
p2	θ2	v21	v22	v23	v24	v2*
p3	θ3	v31	v32	v33	v34	v3*
p4	θ4	v41	v42	v43	v44	v4*
	EMV	E[A1]	E[A2]	E[A3]	E[A4]	E[v*]
	CE	CE[A1]	CE[A2]	CE[A3]	CE[A4]	CE[v*]

$EVPI = E[v^*] - Max_j \{ E[Aj] \}$

$CEVPI = CE[v^*] - Max_j \{ CE[Aj] \}$

EVSI CONCEPT EXTENSION

Extensive form analysis (SI=Sample Information)

$EVSI = E[v|SI] - Max_j \{ E[Aj] \}$

$CEVSI = CE[v|SI] - Max_j \{ CE[Aj] \}$

THEOREM

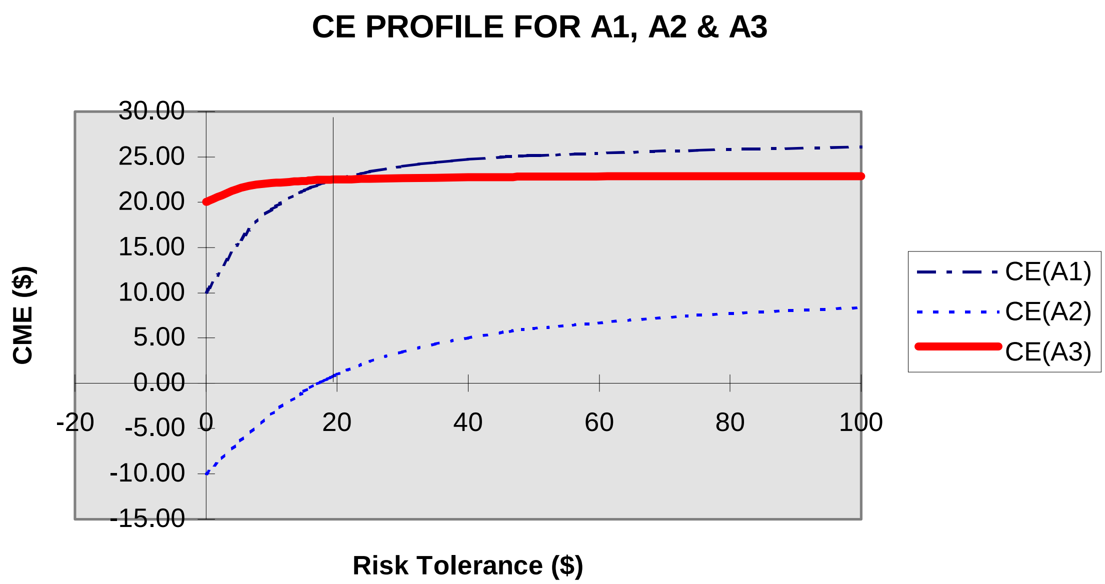
$0 \leq EVSI \leq EVPI$

$0 \leq CEVSI_\tau \leq CEVPI_\tau \quad \text{all } \tau > 0$

CE-DOMINANCE

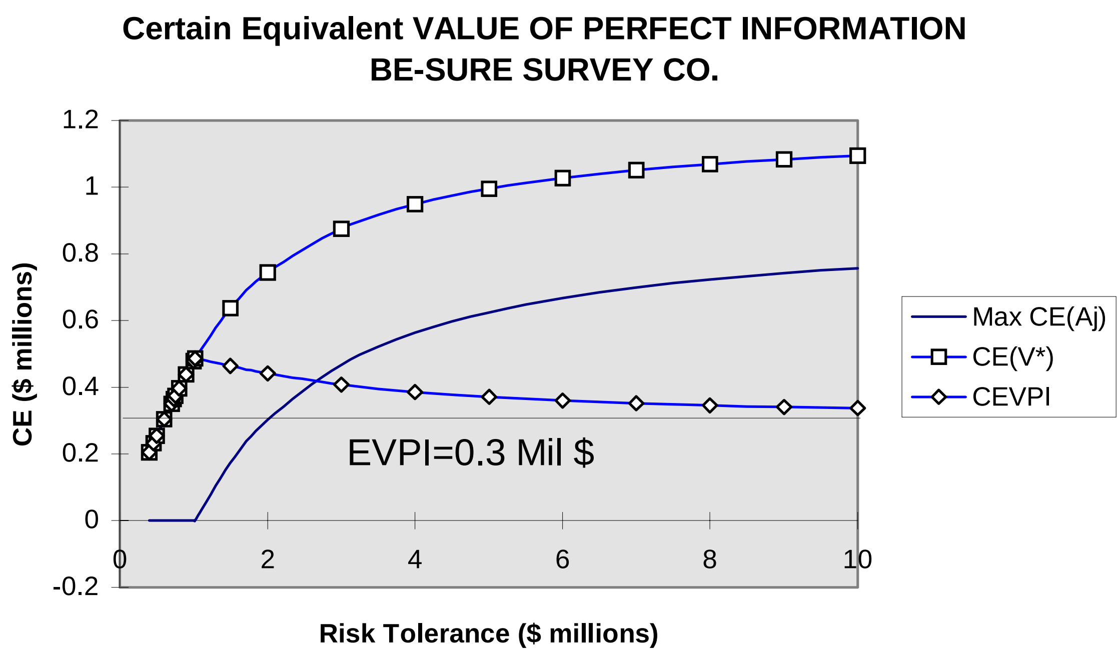
L is CE-Dominated by $\{A_1,...,A_n\}$ if

$CE_\tau(L) < MAX_i(CE_\tau(A_i))$ for all τ

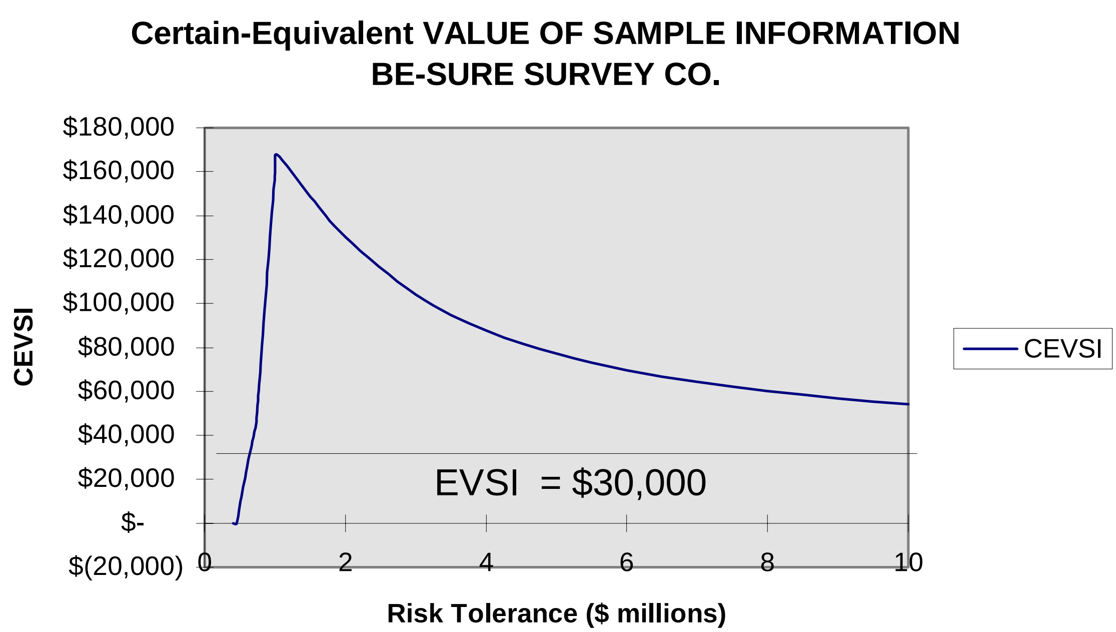


A2 is CE-dominated by A1 and also by A3

A2 is CE-dominated by {A1,A3}



CEVPI rises to over 1.5 times the value of EVPI.



CEVSI rises to over 5.5 times the value of EVSI.

COMPARING NORMAL GAMBLES

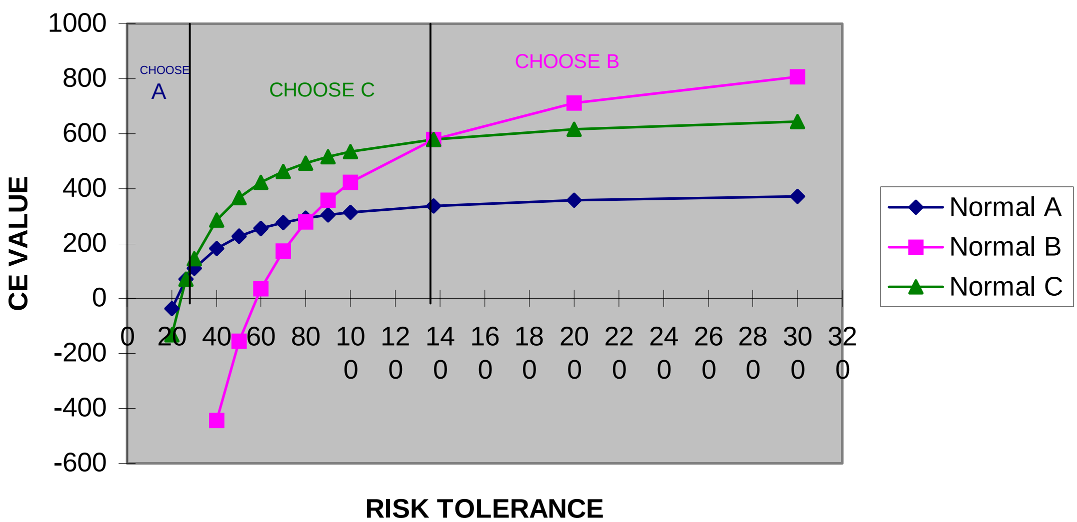
$$CE_{\tau}(\mu,\sigma) = \mu - 0.5\ \sigma^2/\tau$$

GAMBLE A: Normal with Mean $400, Standard Deviation $132

H. GAMBLE B: Normal with Mean $$1,000, Standard Deviation $340

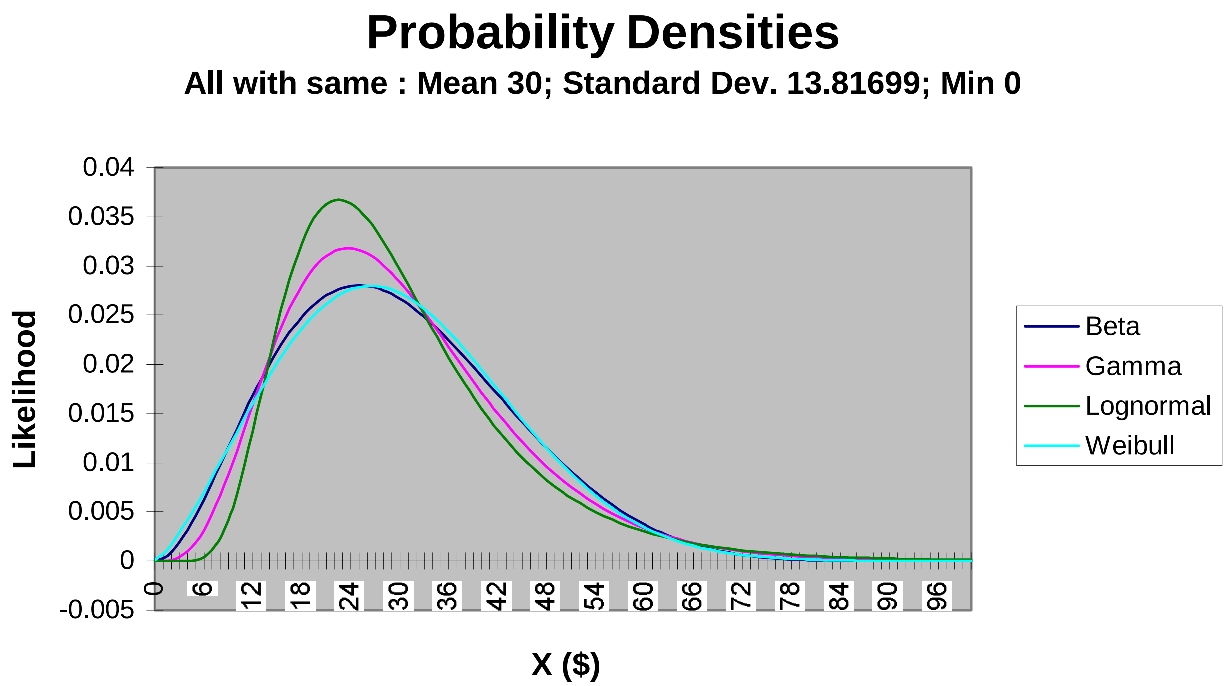
GAMBLE C: Split investment 50-50 between A & B:
Mean $700, Standard Deviation $183.26

Figure 1
Normal CE VALUES vs RISK TOLERANCE

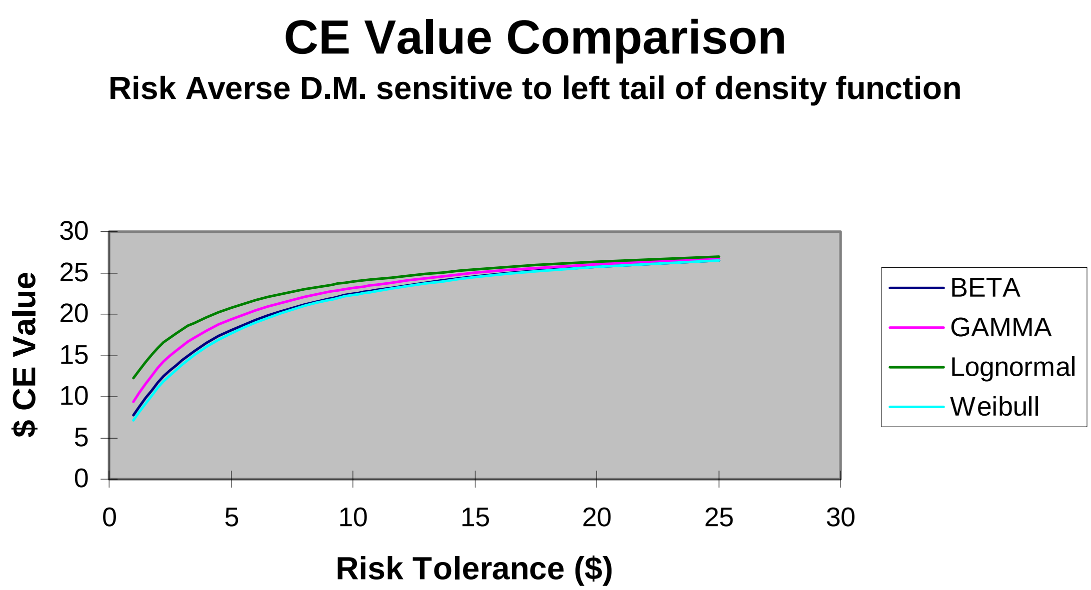


CE COMPARISON CHART

Consider four distributions having the same mean, same variance, and same downside, from the beta, gamma, lognormal, and Weibull families.



Ordering by left tail "thickness": Lognormal, Gamma, Beta, Weibull



EXPONENTIAL UTILITY FUNCTION ADVANTAGES

I. THEORETICAL ADVANTAGES

* Value Additivity Property

* EVPI Concept Extension

* EVSI Concept Extension

* Wealth Dependent Tolerances

II. COMPUTATIONAL ADVANTAGES

* Closed Form CE-functions

* Series Expansion CE-functions

* Alternate Tolerance Estimation Methods

  - Triple-or-Nothing Bet
  - Least Square CE-fit
  - Parametric Backtest Studies

CERTAIN EQUIVANCE FUNCTIONS
NOW KNOWN FOR THESE IMPORTANT
DISTRIBUTION FAMILIES

DISCRETE DISTRIBUTIONS (8)	CONTINUOUS DISTRIBUTIONS (15)
Binomial Finite Discrete Geometric Hyper-Geometric Negative Binomial Pascal Poisson Uniform (discrete)	Beta Chi-Square Erlang Exponential Gamma Histogram Inverse Gamma Laplace Logistic LogNormal Normal Rayleigh Triangular Uniform (continuous) Weibull

- Available at www.matpro.com/cevalues/cecalc.html NOW!
- Available at www.matpro.com/cevalues/cecalc.html in 2 weeks!
- Available in JAI paper NOW!

<HTML>

```
<HEAD>
<TITLE>Ron's JavaScript(tm) Triangular Value Calculator</TITLE>
<SCRIPT LANGUAGE="JAVASCRIPT">
<!-- hide this script tag's contents from old browsers
function triangle_ce(a, m, b, tau) {a = parseFloat(a); m=parseFloat(m);
b=parseFloat(b); tau=parseFloat(tau);
  var ab = (b-a)/tau;
 var am = (m-a)/tau;
 var mb = (b-m)/tau;
var expam = Math.exp(-am); var expmb = Math.exp(-mb);
var arg = (2/ab)*((1-expam)/am - expam*(1-expmb)/mb)
var ce_val= a - tau*Math.log(arg);
return ce_val;
}
function computeform(form) {
       if (checkform(form)) {
         form_a=form.amin.value;
         form_m=form.mmid.value;
         form_b=form.bmax.value;
         form_tol=form.risktol.value;

          low=Math.round(triangle_ce(form_a, form_m,
form_b,0.9*form_tol)*100);
          form.low_val.value=low/100;

          ce=Math.round(triangle_ce(form_a, form_m,
form_b,form_tol)*100);
          form.ce_val.value=ce/100;

          high=Math.round(triangle_ce(form_a, form_m,
form_b,1.1*form_tol)*100);
          form.high_val.value=high/100;
       }
       return;
}
```

ASSET ALLOCATION MODELS for ASYMMETRIC RETURN DISTRUBTIONS

Allocate assets to MAXIMIZE

CERTAINTY EQUIVALENT VALUE

Under Normal forecast:

Maximize $\sum r_i x_i - (1/2)\, X^T Q X/\tau$

S.T. $\sum x_i = b$ $\quad X \geq 0$

Under Gamma forecast:

$$Maximize \quad P_\delta + \tau \frac{(P_\mu - P_\delta)^2}{P_\upsilon} \ln[1 + \frac{P_\upsilon}{\tau(P_\mu - P_\delta)}]$$

S.T. $\sum x_i = b$ $\quad X \geq 0$

$\sum \delta_i x_i = P\delta$ **(Portfolio Min)**

$\sum r_i x_i = P\mu$ **(Portfolio Mean)**

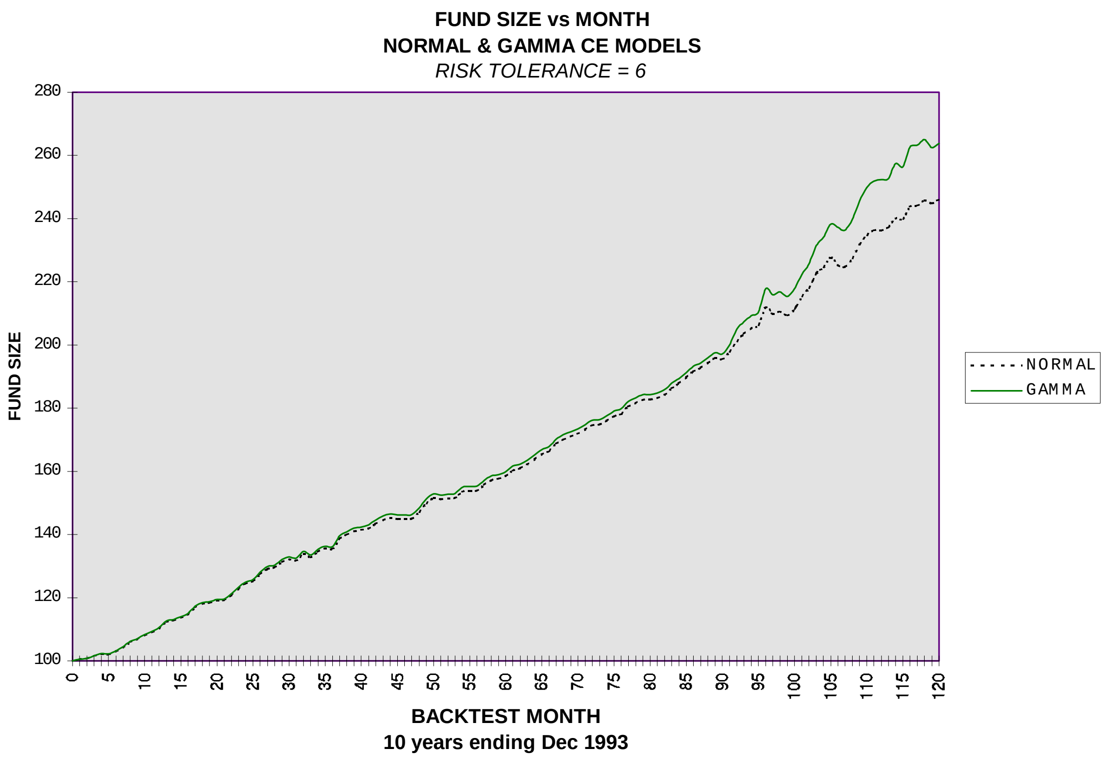
$X^T Q X = P\upsilon$ **(Portfolio Variance)**

PORTFOLIO BACKTEST RESULTS

Normal CE-criterion: CE = $\mu - \sigma^2/(2\tau)$

Gamma CE-criterion: CE = $$\underline{x} + \tau\frac{(\mu-\underline{x})^2}{\sigma^2}\ln\left(1+\frac{\sigma^2}{\tau(\mu-\underline{x})}\right)$$

Results for $\tau = 6$



R.E. Davis, "An Empirical Back Test of the Portfolio Gamma Model for Optimal Asset Allocation", Vol 4, ***Advances in Mathematical Programming and Financial Planning***, JAI Press, 1995.

PORTFOLIO OPTIMIZATION WITH SCENARIOS

Assume m scenarios with returns and covariances given by (r_1, C_1) and probability of occurance p_i.

Then each portfolio allocation x leads to m different mean, variance outcomes as follows:

$$\mu_i = r_i x \text{ and } \sigma_i^2 = x^T C_i x$$

I. **The CE for each normal outcome is therefore given by**

$$CE_i = \mu_i – 0.5\ \sigma_i^2/\tau$$

Since this value occurs with probability p_i, the CE for the overall scenario set is just

$$\text{scenarioCE}_\tau = -\tau \ln(\Sigma_i\ p_i \exp(-(\mu_i – 0.5\ \sigma_i^2/\tau)/\tau)$$

Now x may be optimized with respect to this criterion.

PORTFOLIO SCENARIO OPTIMIZATION

Table 6

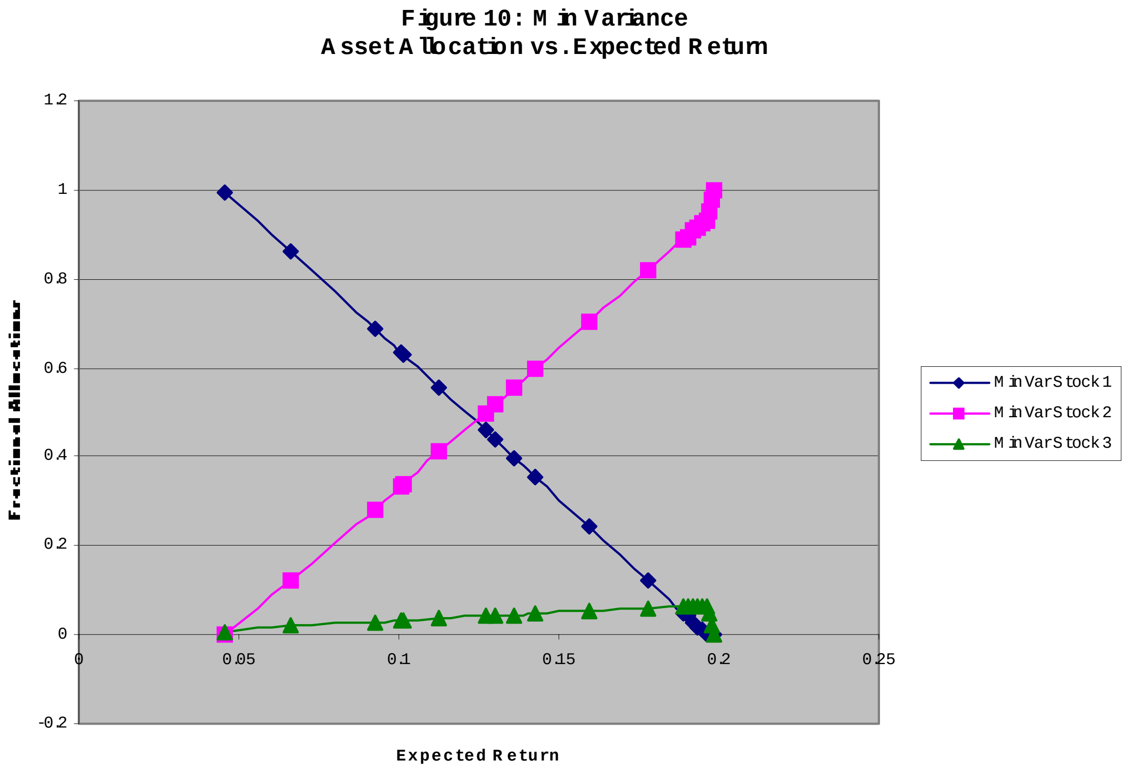
Scenario #k	**Stock 1**	**Stock 2**	**Stock 3**
1	-.071	.144	.169
2	.056	.107	-.035
3	.038	.321	.133
4	.089	.305	.732
5	.090	.195	.021
6	.083	.390	.131
7	.035	-.072	.006
Standard dev	.02/k	.15/k	.08/k

Correlation Assumptions:

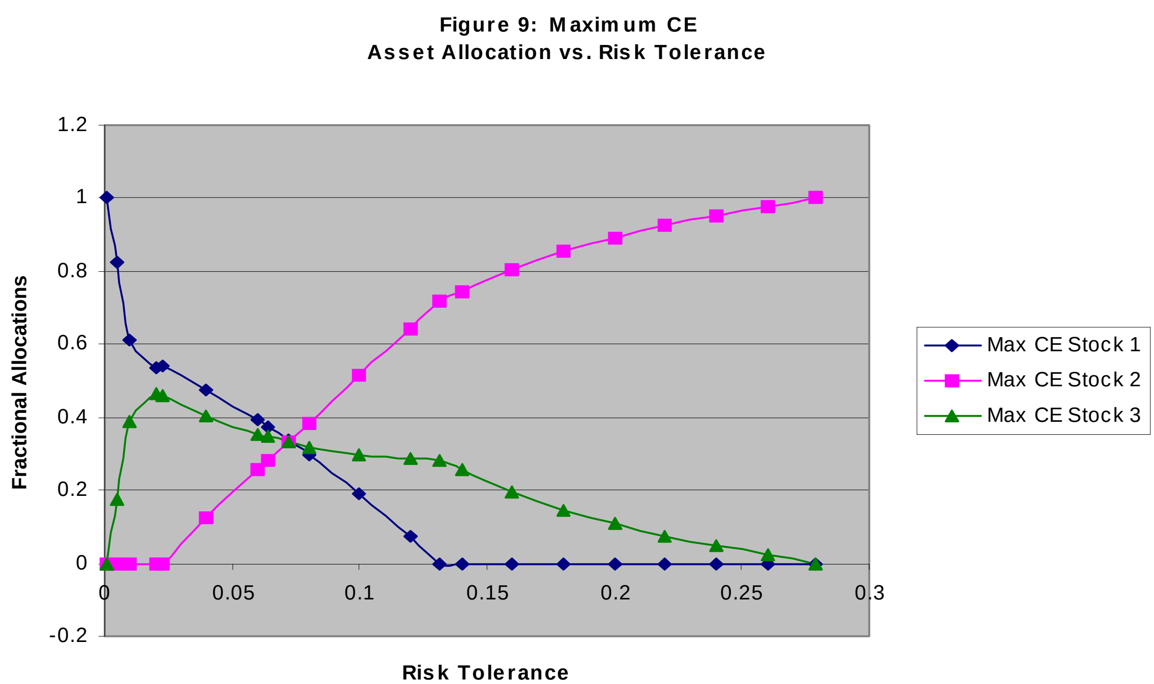
$\rho_{12} = 0.6$, $\rho_{13} = 0.4$, $\rho_{23} = 0.7$

Equally likely scenarios: $p_i = 1/7$ $(i=1,\ldots,7)$

Allocations as a function of Risk Tolerance: Min Variance criterion.



Allocations as a function of Risk Tolerance: Max scenarioCE criterion.



CONCLUSIONS

1. Virtually all commonly used probability distributions admit of CE-Value functions obtained

- In closed form, or
- As a series expansion, or
- through numerical integration

2. Value additivity enables the value of information concepts EVPI and EVSI to be extended to the risk averse case, yielding $CEVPI_\tau$ and $CEVSI_\tau$.

3. The CE Value of Perfect and Sample Information may exceed EVPI and EVSI respectively, in some cases by a wide margin. A risk averse D.M. may value sample information much more highly than a risk neutral one does, hence EVSI and EVPI may NOT be taken as upper limits on amounts spent on sample information.

4. For asymmetric return distributions and for scenario optimizations,

$$\text{Mean-Variance efficiency} \neq \text{CE Maximality}$$

The Mean-Variance Efficient Frontier is replaced with the CE-Maximality frontier as a set from which to choose the 'optimal' solution.

5. Backtest results indicate that CE-Maximal solutions MAY perform better than Mean-Variance efficient ones in actual practice.

6. Entire "callable library" should be available by September, 1998. Keep watching www.matpro.com